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A COMMON FIXED POINT THEOREM IN MENGER SPACE WITH COMPATIBLE MAPPINGS OF TYPE (P)

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Abstract

The purpose of paper is to introduce the notion of compatible mappings of type (P) in Menger space and to obtain a common fixed point theorem in this space by using this mapping with example. Our result generalizes and improves some known similar results in literature.

1. Introduction

The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research during last three decades. The notion of distance

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was created by M. Frechet [3] in 1906. In 1914, Hausdroff [4] inaugurated the distance notion by Metric Space name. Austrian Mathematician Karl's Menger in 1942 introduced Menger Space [9] as one of important generalizations of Metric Space in Probability version. The study of this space was expanded rapidly due to pioneer work of B. Schweizer and A. Skalar [15], [12] in 1960. And it became active research area for researcher when V. M. Sehgal and A. T. Barucha Reid [13] introduced a contraction mapping in Probabilistic Metric Space as generalization of Banach Contraction Principle given by S. Banach [1] in metric space and established fixed point theorems in Menger Probabilistic Metric Space. Also, the survey work on contractions in probabilistic metric space has been studied by A. K. Chaudhary and K. Jha [6] in 2019.
The compatible mapping in probabilistic metric space was introduced by S.N.Mishra [10] in 1991. After that so many compatible mappings has been developed by researchers in Probabilistic Metric Space. HK Pathak et.al. [11] in 1996 had introduced the concept of compatible mapping of type (P) in metric space. In this paper, we extends compatible mapping of type (P) in Menger space and obtain a common fixed point theorem with suitable example which generalizes some well known results in literature.

2. Preliminaries

Definition 2.1 [15] : A function $F:\mathbb{R}\to\mathbb{R}^+$ is said to be *distribution function* if it is

(i) Non-decreasing function

(ii) left continuous

(iii) $Inf\{F(x):x\in\mathbb{R}\}=0$ and $sup\{F(x):x\in\mathbb{R}\}=1$.

Here, we denote the set of all distribution function by L and H denotes distribution function, called Heavy Side Function, defined as:

$$H(x)=\begin{cases}0, & if \quad x\leq 0\\ 1, & if \quad x>0\end{cases}$$

Definition 2.2 [15] : Let K be a non-empty set and $F:K\times K\to L$ is a distribution function. Then, a pair (K,F) is said to be Probabilistic Metric Space (briefly, PM-Space) if the distribution function $F(p,q)$, where $(p,mq)\in K\times K$, also denoted by $F_{p,q}$ satisfy following conditions:

PM(1) : $F_{p,q}(x) = 1$ for every $x > 0$ if and only if $?p = q$

PM(2) : $F_{p,q}(0) = 0$ for every $p, q \in K$

PM(3) : $F_{p,q}(x) = F_{v,u}(x)$ for every $p, q \in K$

PM(4) : $F_{p,q}(x + y) = 1$ if and only if $F_{p,r}(x) = 1$ and $F_{r,q}(y) = 1$.

Here, $F_{p,q}(x)$ represents the value of $F_{p,q}$ at $\in \mathbb{R}$

Definition 2.3 [5] : A function $t : [0,1] \times [0,1] \to [0,1]$ is called *triangular norm* (T-norm) if it satisfies the following conditions:

$\mathbf{T_1}$: $t(0,0) = 0$ and $t(a,1) = a$ for all $a \in [0,1]$

$\mathbf{T_2}$: $T(a,b) = t(b,a)$ for all $a, b \in [0,1]$

$\mathbf{T_3}$: $if\ a \leq c, b \leq d$ then $t(a,b) \leq t(c,d)$ and

$\mathbf{T_4}$: $t(t(a,b),c) = t(a,t(b,c))$.

Definition 2.4 [15] : A triplet (K, F, t) is said to be *Menger Space or Menger Probabilistic Metric Space* where (K, F) is PM space and t is T-norm which satisfies the condition:

PM 5 : $F_{p,q}(x + y) \geq t(F_{p,r}(x), F_{r,q}(y))$ for every $p, q, r \in K$ and $x, y \in \mathbb{R} > 0$.

Remark 2.1 : If (K, d) be metric space then metric d induces a distribution function F defined by $F_{p,q}(t) = H(t - d(p,q))$ and if f is contraction and $d(fp, fq) \leq kd(p,q)$ in Metric Space, then, in PM Space

$$F_{fp,fq}(kt) \geq F_{p,q}(t)$$

and when if $d(p,q) < t$ then $F_{p,q}(t) > 1 - t$.

Also, $F_{fp,fq}(kt) \geq F_{p,q}(t)$ whenever $F_{p,q}(t) > 1 - t$.

Definition 2.5 [6] : A mapping $Q : K \to K$ in Menger Space (K, F, t), is said to be *Continuous* at a point $p \in K$ if for every. $\epsilon > 0$ and $\lambda > 0$, there exist $\epsilon_1 > 0$ and $\lambda_1 > 0$ such that if $F_{p,q}(\epsilon_1) > 1 - \lambda_1$ then $F_{Qp,Qq}(\epsilon) > 1 - \lambda$.

Definition 2.6 [6] : Let (K, F, t) be a Menger Space and t be a continuous T-norm. Then,

(a) A sequence $\{k_n\}$ in K is said to be converge to a point k in K if and only if for every $\epsilon > 0$ and $\lambda > 0$, there exist an integer $N = N(\epsilon, \lambda)$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \geq N$. We write, $\lim\limits_{n\to\infty} k_n = k$.

(b) A sequence $\{k_n\}$ in K is called a Cauchy Sequence if for every $\epsilon > 0$ and $\lambda > 0$, there exist an integer $N = N(\epsilon, \lambda) > 0$ such that $F_{k_n,k_m}(\epsilon) > 1 - \lambda$ for all $m, n \geq N$.

(c) A Menger Space (K, F, t) is said to be Complete if every Cauchy sequence in K Converges to a point in K.

In 1991 S. N. Mishra [10] have introduced following Compatible in Menger Space as the extension of compatible mapping in metric space introduced by G. Jungck [7] in 1986.

Definition 2.7 : Two mappings $Q, R : K \to K$ are said to be *Compatible* in Menger Space (K, F, t) iff $\lim_{n\to\infty} F_{QRk_n,\ RQ}(x) = 1$ for all $x > 0$, whenever $\{k_n\}$ in K such that $\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Rk_n = z$ for some z in K.

Definition 2.8 [2] : Two mappings $Q, R : K \to K$ are said to be *Compatible of type (A)* in Menger Space (k, f, T) iff $\lim_{n\to\infty} F_{QR,RRk_n}(x) = 1$ and $\lim_{n\to\infty} F_{RQk_n,QQk_n}(x) = 1$ for all $x > 0$, whenever $\{k_n\}$ is a sequence in K such that $\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Rk_n = k$ for some k in K.

H. K. Pathak et.al [11] in 1996 had introduced the concept of compatible mapping of type (P) in metric space. As an extension of this type mapping, we have introduced the following compatible mapping of type (P) in Menger space with example as follows:

Definition 2.9 : Two mappings $Q, R : K \to K$ are said to be *Compatible Mapping of type (P)* in Menger Space (K, F, t) iff $\lim_{n\to\infty} F_{QQk_n,RRk_n}(x) = 1\ \forall\ x > 0$ whenever $\{k_n\}$ is a sequence in K such that $\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Rk_n = k$ for some k in K.

Example : Let (K, d) be metric space where $K = 0, 2]$ with usual metric $d(x, y) = |x-y|$ and (K, F) be PM space with

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & if \quad t > 0 \\ & \qquad\qquad \text{for all} x, y \in K \\ 0, & if \quad t = 0. \end{cases}$$

Defining Q and R as:

$$Q(x) = \begin{cases} 1, & for \quad x \in [0, 1) \\ 2 - x, & for \quad x \in [1, 2] \end{cases} \quad \text{and} \quad \begin{cases} 1, & for \quad x \in [0, 1) \\ x, & for \quad x \in [1, 2] \end{cases}$$

Taking $\{k_n\}$ in K where $k_n = 1 + \frac{1}{n}, n \in N$

$$\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Q\left(1 + \frac{1}{n}\right) = 2 - \left(1 + \frac{1}{n}\right) = 1 = t$$

$$\lim_{n\to\infty} Rk_n = \lim_{n\to\infty} R\left(1 + \frac{1}{n}\right) = \left(1 + \frac{1}{n}\right) = 1 = t$$

$$QQk_n = Q(Q\left(\left(1+\frac{1}{n}\right)\right) = Q(1) = 1+t.$$

Since, $RRk_n = R(R\left(\left(1+\frac{1}{n}\right)\right) = R(1) = 1 = t$

Therefore, $\lim_{n\to\infty} F_{QQk_n},\ {}_{RRk_n}(x) = \lim_{n\to\infty} F_{1,1}(x) = \frac{x}{x+|1-1|} = 1$ for all $x > 0$.

Hence, (Q, R) is compatible of type P in PM Space.

It is important to note that if $Q, R : K \to K$ be compatible mapping of type (A) and one of Q and R is continuous then Q and R are compatible of type P.

Theorem 2.1 [2] : Let $K, F, t)$ be Menger space with the continuous $T-norm$ t and $Q : K \to K$. Then, Q is continuous at a point $k \in K$ if and only if for every sequence $\{k_n\}$ in K converging to k, the sequence $\{Qk_n\}$ converges to the point Qk. i.e. if $\{k_n\} \to k$ then it implies $\{Qk_n\} \to Qk$.

Proposition 2.1 [14] : In Menger Space $K, F, t)$, if $t(k, k) \geq k$ for all $k \in [0, 1]$ then $t(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$.

Lemma 2.1 [14] : Let $(K, F.t)$ be a Menger space. If there exists $k \in (01)$ such that for $p, q \in K, F_{p,q}(kx) \geq F_{p,q}(x)$ then $p = a$.

Proof : Since, $F_{p,q}(kx) \geq F_{p,q}(X)$. So, we have

$$F_{p,q}(kx) \geq F_{p,q}(k^{-1}x).$$

By repeated application of above in equality, we get

$$F_{p,q}(x) \geq F_{p,q}(k^{-1}x) \geq F_{p,q}(k^{-2}x) \geq \cdots \geq F_{p,q}(l^{-m}x) \cdots m \in N$$

which implies 1 as $n \to \infty$.

Hence, $F_{p,q}(x) = 1\ \forall\ \ x > 0$ and we get $p = q$.

We need following propositions for the establishment of our main results in Menger space.

Proposition 2.2 : Let $K, F, t)$ be a Manger Space such that the T-norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $G, H : K \to K$ be mappings. If G and .. are compatible mapping of type P and $Gz = Hz$ for some $z \in K$, then, $GGZ = GHz = HGz = HHz$.

Proof : Suppose $\{k_n\}$ is a sequence in K defined by $k_n = z$ where $n = 1, 2, 3, \cdots$ for some $z \in K$ and $Gz = Hz$. Then we have $Gk_n, Hk_n \to Gz$ as ?$n \to \infty$.

Since, G and H are compatible of type P, then for every $\epsilon > 0$

$$F_{GGz,HHz}(\epsilon) = \lim_{n\to\infty} F_{GGK_n,HHK_n}(\epsilon) = 1.$$

$\therefore \quad GGz = HHz$. But $Gz = Hz$ implies $GGz = GHz = HGz = HHz$.

Proposition 2.3 : Let (K, F, t) be a Manger Space such that the T-norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $G, H : K \to K$ be mappings. Let G and H are compatible mappings of type P and

$$\lim_{n\to\infty} Gk_n = \lim_{n\to\infty} Hk_n = z \text{ for some } z \in K.$$

Then, we have,

(i) $\lim_{n\to\infty} HHk_n = Gz$ kif G is continuous at z,

(ii) $\lim_{n\to\infty} GGk_n = Hz$ if H is continuous at z,

(iii) $GHz = HGz$ and $Gz = Hz$ if G and H are continuous at z.

Proof (i) : Suppose that G is continuous at z. Since, we have,

$$\lim_{n\to\infty} gk_n = \lim_{n\to\infty} Hk_n = z \text{ for some } z \in K.$$

So $\lim_{n\to\infty} GGk_n = Gz$.

Again, since G and H are compatible of type P.

Therefore, $\lim_{n\to\infty} F_{GGk_n,HHk_n}(\epsilon) = 1$ for all $\epsilon > 0$. So

$$\begin{aligned} F_{HHk_n,Gz}(\epsilon) &\geq t(F_{HHk_n,GGk_n}(\epsilon/2) \\ &\geq t(1, F_{Gz,Gz}(\epsilon/2)) \\ &\geq t(1, 1) \end{aligned}$$

$$F_{HHk_n,Gz}(\epsilon) = 1.$$

So, $\lim_{n\to\infty} HHk_n = Gz$.

(ii) We may prove (ii) as we prove (i)

(iii) Suppose that $G, H : K \to K$ are continuous at z.

Since, $\lim_{n\to\infty} Hk_n = z$ and G is continuous at z.

So, by Proposition 2.3 (i) $HHk_n \to Gz$, as $n \to \infty$.

On the other hand, since, $\lim_{n\to\infty} Hk_n = z$ and H is also continuousat z.

$$\therefore \quad \lim_{n\to iny} HHk_n = Hz.$$

Thus, we have $Gz = Hz$ by uniqueness of limit and so by Proposition 2.2, $HGz = GHz$. This completes the proof.

Lemma 2.2 [14] : Let $\{k_n\}$ be a sequence in Menger space (K, F, t), where t is continuous T-norma and $t(x, x) \geq x$ for all $x \in [0, 1]$. If there exists a constant $k \in [0, 1]$ such that

$$F_{k_{n},k_{n+1}}(kx) \geq F_{k_{n-1},k_{n}}(x)$$

for all $x > 0$ and $n \in N$, then $\{k_n\}$ is a Cauchy sequence in K.

3 . Main Results

Now we prove our main theorem for compatible mappings of type (P) in complete Menger space:

Theorem 3.1 : Let (K, F, t) be a complete Menger Space with $(x, y) \min\{x, y\}$ for all $x, y \in [0, 1]$ and $Q, S, R, T : K \to K$ be mappings such that

(3.1) $Q(K) \subset T(K)$ and $S(K) \subset R(K)$

(3.2) the pairs (Q, R) and (S, T) are compatible of type P.

(3.3) One of Q, S, R, T be continuous.

(3.4) there exist a constant $\epsilon \in (0, 1)$ such that

$$F_{Qx,Sy}(kt) \geq \min\{F_{Rx,Qx}(t).F_{Ty,Qx}(\alpha t), F_{Rx,Sy}(2-\alpha)t), F_{RxTy}(t)\}$$

for all $x, y \in K, \alpha \in (0, 2)$ and $t > 0$.

Then Q, R, S, T have a unique common fixed point in K.

Proof : Consider $u_0 \in K$. Since $Q(K) \subset T(K)$. So, there exists a point u_1 in K. Such that $Qu_0 = Tu_1 = v_0$. Again, since $S(K) \subset R(K)$. So, for u_1, we may choose u_2 in K such that $Su_1 = Ru_2 = v_1$ and so on.

And inductively, we may construct sequence $\{u_n\}$ and $\{v_n\}$ in K such that

$$\begin{aligned} Qu_{2n} &= Tu_{2n+1} = v_{2n} \\ Su_{2n+1} &= Ru_{2n+2} = v_{2n+1}, \quad for \quad n = 0, 1, 2, . \end{aligned}$$

Putting $x = u_{2n}$ and$y = u_{2n+1}$ for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$ in (3.4), we get

$$F_{Qu_{2n},Su_{2n+1}} \geq \min \left\{ \begin{array}{l} F_{Ru_{2n},Qu_{2n}}(t), F_{Tu_{2n+1},Su_{2n+1}}(t), F_{Tu_{2n+1},Qu_{2n}}(2-q)t), \\ F_{Ru_{2n},Su_{2n+1}}((1+q)t), F_{Ru_{2n},Tu_{2n+1}}(t) \end{array} \right\}$$

$$\begin{aligned} & F_{v_{2n},v_{2n+1}}(kt) \\ \geq \;& \min\{F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n},v_{2n}}((1-q)t), F_{v_{2n-1},v_{2n+1}}((1+q)t).F_{v_{2n-1},v_{2n}}(t)\} \\ \geq \;& \min\{F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(qt)\} \\ \geq \;& \min\{F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n},v_{2n+1}}(qt)\}. \end{aligned}$$

As $q \to 1$, we obtain

$$\begin{aligned} F_{v_{2n},v_{2n+1}}(kt) &\geq \min\{F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n},v_{2n+1}}(t)\} \\ &= \min\{F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t)\}. \end{aligned}$$

Hence $f_{v_{2n},v_{2n+1}}(kt) \geq \min\{F_{v_{2n-1},v_{2n}}(t), mF_{v_{2n},v_{2n+1}}(t)\}$.
i.e. $F_{v_{2n},v_{2n+1}}(kt) \geq F_{v_{2n-1},v_{2n}}(t)$.
Similarly, also we obtain

$$F_{v_{2n+1},v_{2n+2}}(kt) \geq F_{v_{2n},v_{2n+1}}(t).$$

Therefore, for every $n \in N$,

$$F_{v_n,v_{n+1}}(kt) \geq F_{v_{n-1},v_n}(t)/$$

So, by Lemma 2.2 $\{v_n\}$ is Cauchy sequence in K.
Since, the Menger space (K, F, t) is complete, $\{v_n\}$ converges to a point z in K and the subsequences $\{Q_{u_{2n}}\}, \{S_{u_{2n+1}}\}, \{R_{u_{2n}}\}, \{T_{u_{2n+1}}\}$of $\{v_n\}$ also converts to z.
Now, suppose that T is continuous, since S and T are complatible of type (P), then by Proposition 2.3, $SS_{u_{2n+1}}, TS_{u_{2n+1}} \to TZ$ as $n \to \infty$. Putting $x = u_{2n}$ and $y = S_{u_{2n+1}}$ in (1.4), we get

$$\begin{aligned} & F_{Qu_{2n},SSu_{2n+1}}(kt) \\ \geq \;& \min \left\{ \begin{array}{r} F_{Ru_{2n-1},Qu_{2n}}(t), F_{Tsu_{2n+1},SSu_{2n+1}}(t), F_{TSu_{2n+1},Qu_{2n}}((\alpha)t), \\ F_{Ru_{2n},SSu_{2n+1}}((2-\alpha)t), F_{Ru_{2n},TSu_{2n+1}}(t) \end{array} \right\} \end{aligned}$$

Taking $n \to \infty$, we have

$$F_{z,Tz}(kt) \geq \min\{F_{z,z}(t), F_{Tz,Tz}(t), F_{Tz,z}(\alpha t), F_{z,Tz}((2-\alpha)t), F_{z,Tz}(t)\}.$$

Letting $\alpha = 1 - q$ with $q \in (0,1)$ then

$$F_{z,Tz}(kt) \geq \min\{F_{Tz,z}((1-q)t), F_{z,Tz}(t)((2-(1-q))t), F_{z,Tz}(t)\}.$$

$$\begin{aligned} F_{z,Tz}(kt) &\geq \min\{F_{Tz,z}((1-q)t), F_{z,Tz}((1+q)t), F_{z,Tz}(t)\} \\ &\geq \min\{F_{Tz,Tz}((1-q+1+q)t), F_{z,Tz}(t)\} \\ &\geq \min\{F_{z,Tz}(t)\}. \end{aligned}$$

$\therefore \quad F_{z,Tz}(kt) \geq F_{z,Tz}(t)E$ which implies $z = Tz$ by Lemma 2.1.
Simiarly, replacing x by u_{2n} and y by z in (3.4), we have

$$F_{Qu_{2n}Sz}(kt) ge \min\{F_{Ru_{2n},Qu_{2n}}(t), F_{Tz,Sz}(t), F_{Tz,Qu_{2n}}(\alpha t), F_{Ru_{2n},Sz}((2-\alpha)t), F_{Ru_{2n},Tz}(t)\}.$$

Taking $n \to \infty$

$$\begin{aligned} F_{z,Sz}(kt) &\geq \min\{F_{z,z}(t), F_{z,Sz}(t), F_{z,z}(\alpha t), F_{z,Sz}((2-\alpha)t), F_{z,z}(t)\} \\ &\geq \min\{F_{z,Sz}(t), F_{z,Sz}((2-(1-q)t)\} \\ &\geq \min\{F_{z,Sz}(t), F_{z,Sz}((1+q)t)\} \\ &\geq \in \{F_{z,Sz}(t), F_{z,z}(t), F_{z,Sz}(qt)\} \\ &\geq \min\{F_{z,Sz}(t), F_{z,Sz}(t)\} \quad \text{as} \quad q \to 1. \end{aligned}$$

$\therefore \quad F_{z,Sz}(kt) \geq F_{z,Sz}(t)$.
So, $z = Tz$.
Since $S(K) \subset R(K)$. So, there exists a point w in X such that $Sz = Rw = z$.
By using (3.4), again, we have

$$\begin{aligned} F_{Qw,z}(kt) &\geq \min\{F_{Rw,Qw}(t), F_{Tz,Sz}(t), F_{Tz,Qw}(\alpha t), F_{Rw,Sz}((2-\alpha)t).F_{Rw,Tz}(t)\} \\ &\geq \min\{F_{z,Qw}(t), F_{Tz,z}(t), F_{z,Qw}((1-q)t), F_{Rw,z}((1+q)t), F_{z,Tz}(t)\} \\ &\geq \min\{F_{z,Qw}(t), F_{Tz,z}(t), F_{Qw,z}((1-q)t), F_{Rw,z}((1+q)t), F_{z,Tz}(t)\} \\ &\geq \min\{F_{z,Qw}(t), mF_{z,z}(t), F_{Qw,Rw}((1-q+1+q)t)\} \\ &\geq \min\{F_{z,Qw}(t), mF_{Qw,z}(2t)\} \end{aligned}$$

$\therefore \quad F_{Qw,z}(kt) \geq F_{z,Qw}(t) = F_{Qw,z}(t).$

$\therefore \quad Qw = z.$

Since Q and R are compatible mapping of type (P) and $Qw = Rw = z$, by Proposition 2.2, we have for every $\epsilon > 0$.

$F_{QRw,RRw}(\epsilon) = 1$ and hence $Qz = QRw + RRw + Rz$.

Finally, by (3.4), we have (here $x = z, z = Sz$)

$$\begin{aligned} F_{Qz,z}(kt) &= F_{Qz,Sz}(kt) \geq \min\{F_{Rz,Qz}(t), F_{Tz,z}(t), F_{Tz,Qz}(\alpha t), F_{Rz,z}((2-\alpha)t), F_{Rz,Tz}(t)\} \\ &\geq \min\{F_{Qz,Qz}(t), F_{z,z}(t), F_{z,Qz}(\alpha t), F_{Qz,z}((2-\alpha)t), F_{Qz,z}(t)\} \\ &\geq \min\{F_{Qz,z}(\alpha, t), F_{z,Qz}((2-\alpha)t), F_{Qz,z}(t)\} \\ &\geq \min\{F_{Qz,Qz}(\alpha t + (2-\alpha)t), F_{Qz,z}(t)\}. \end{aligned}$$

Or, $F_{Qz,z}(kt) \geq F_{Qz,z}(t)$.

$\therefore \quad Qz = z.$

Hence, $Qz = Sz = Rz = Tz = z$.

This is, z is common fixed point of given mappings Q, R, S and T.

Uniqueness : Suppose z_1 be other point in K such that

$$z_1 = Qz_1 = Sz_1 = Rz_1 = Tz_1.$$

Putting $x = z$ and $y = z_1$, $\alpha = 1$ in (3.4), we get

$$\begin{aligned} F_{Qz,Sz_1}(kt) &= F_{z,z_1}(kt) \geq \min\{F_{Rz,Qz}(t), F_{Tz_1Sz_1}(t), F_{Tz_1.Qz}t).F_{Rz,Sz_1}(t), F_{Rz,Tz_1}(t)\} \\ F_{z,z_1}(kt) &\geq \min\{F_{z,z_1}(t), mF_{z,z}(t)\}. \end{aligned}$$

Or, $F_{z,z_1}(kt) \geq F_{z,z_1}(t)$.

By Lemma 2.1, $z = z_1$.

Hence, E$z = Qz = Sz = Rz = Tz$ and z is unique in K.

This completes the proof.

We have following example.

Example 3.1 : Let (K, F, t) be a complete Menger Space with $t(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$, where $K = [1, 10]$ with metric d defined by $d(x, y) = |x - y|$and F is defined by

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{it } \ t > 0 \\ & \qquad \text{for all } \ x, y \in K \\ 0, & \text{if } \ t = 0. \end{cases}$$

Define $Q, S, R, T : K \to K$ as below :

$$Q(x) = \begin{cases} 1 & \text{for } x \leq 4 \\ 2, & \text{for } x > 4 \end{cases} \quad \text{and} \quad S(x) = \begin{cases} 1 & \text{for } x \leq 5 \\ 2, & \text{for } x > 5 \end{cases}$$

$R(x) = Tx = x$ for all $x \in K$.
Taking $\{k_n\}$ in K where $k_n = 1 + \frac{1}{n}, n \in N$. Then Q, S, R and T satisfy all the conditioins of the above theorem 3.1 and have a unique common fixed point $x = 1$.
In the Theorem 3.1, if we take $Q = S, T = R$, then we have following result.
Corollary 3.1 : Let Q and R be self maps in Complete Menger space (K, F, t) with continuous $t(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$ satisfying the following conditioins:L

(i) $Q(K) \subset R(K)$

(ii) (Q, R) be compatible of type P.

(iii) R be continuous.

(3.4) there exists a constant $k \in (0, 1)$ such that

$$F_{Qx,Qy}(kt) \geq \min\{F_{Rx,Qx}(t, F_{Ry,Qy}(t), mF_{Ry,Qx}(\alpha t), F_{Rx,Qy}((2-\alpha)t), F_{Rx,Ry}(t)\}$$

for all $x, y \in K, \alpha \in (0, 2)$ and $t > 0$.
Then Q and R have a unique common fixed point in K.
In the Theorem 3.1, if we take $S = Q = g, R = T = I_K$, an identity mapping on K, then we have following result.
Corollary 3.2 : Let g be self maps in Complete Menger space (K, F, t) with continuous $t(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$. If there exist a constant $k \in (0, 1)$ such that

$$F_{gx,gy}(kt) \geq F_{x,y}(t) = \min\{F_{x,gx}(t), F_{y,gy}(t), f_{y,gx}(\alpha t), F_{x,gy}((2-\alpha)t), F_{x,y}(t)\}$$

for all $x, y \in K, \alpha \in (0, 2)$ and $t > 0$. Then, g has a unique fixed point in K which is the probabilistic version of the Banach contraction theorem established by Sehgal and Bhanarucha Reid [13] in 1972.
Remark : Our result extends and generalizes the results of Jungck et.al [7] in metric space. Also, this result improves other similar results in literature.

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